Strong Nash Equilibria and the Potential Maximizer

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Abstract

A class of non cooperative games characterized by a 'congestion effect' is studied, in which there exists a strong Nash equilibrium, and the set of Nash equilibria, the set of strong Nash equilibria and the set of strategy profiles maximizing the potential function coincide. The structure of the class is investigated and it turns out that this class constitutes a cone. Remarks on strictly strong Nash equilibria and relaxations are provided.

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1 Introduction

In recent years there has been a growing interest in the study of specific classes of non cooperative games for which there exist pure strategy Nash equilibria. One of the pioneering papers is Rosenthal (1973), and Monderer and Shapley (1993) have shown that the existence of a pure equilibrium in these games relies on the possibility of constructing a potential function. In their study of the general class of games which possess this property, Monderer and Shapley have also defined an equilibrium refinement concept, the potential maximizer, and have shown the validity of the fictitious play property (1996). Milchtaich (1994) and Quint and Shubik (1994), on the other hand, have proved the existence of a pure Nash equilibrium for a class of congestion games which, differently from Rosenthal's one, does not in general admit a potential function. Konishi, Le Breton and Weber (1995), considering the same model as Milchtaich, have even shown the existence of a strong Nash equilibrium.

The situation we are going to consider can be described, for example, by the foraging behaviour of a population of identical bees in a field of flowers. In deciding which flower to visit, each insect will take into account the quantity of nectar available and the number of bees already on the flower, because, as is intuitively clear, the more crowded the source of nectar, the less food is available per capita. In economics this kind of problems is studied in the literature on local public goods, where it is common to speak about "anonymous crowding"[1] to describe the negative externality arising from the presence of more than one user of the same facility. Another example is the problem faced by a set of unemployed workers who have to decide where to emigrate to get a job. Different countries are more or less attractive depending on the conditions of the local labour market and, on the other hand, a crowding out effect reduces the appeal of emigrating.

We introduce a variation on the models of Konishi, Le Breton and Weber (Milchtaich) and Rosenthal (Monderer and Shapley) and describe a class of congestion games that provides interesting results. In particular, it will be shown that there exists for each game in this class a strong Nash equilibrium and, moreover, it turns out that the set of Nash equilibria coincides with the set of strong Nash equilibria and the set of potential maximizing strategy profiles.

This paper is structured as follows. In section 2 we investigate the various models mentioned above, clarifying the similarities and differences among them. After that we define a class of games which possesses a strong Nash equilibrium and at the same time admit a potential function in the sense of Monderer and Shapley (1993). In section 3 we anal-

[1] See for example Wooders (1989).

yse the geometric properties of this family of games, showing that it can be represented by a finitely generated cone. In section 4 we state our main theorems concerning the coincidence of equilibrium sets, where the representation of each game as an element of a cone is used. Attention is focused at the computation of the potential. This section is concluded by comments on strictly strong equilibria. The implications of relaxing some of our assumptions underlying the congestion effect are discussed in section 5.

2 Congestion games

The games introduced by Konishi, Le Breton and Weber (KLW, 1995), Milchtaich (1994) and Quint and Shubik (QS, 1994) are rather similar, in the sense that the utility functions of the players are characterized by some kind of "congestion effect". On the other hand however, the slightly different assumptions lead to quite different results, as we are going to see.

Before introducing and comparing the different models, we need to specify some notation. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a normal form game, where $N = \{1, 2, \ldots, n\}$ is the finite set of players, X_i is the finite strategy space of player i and $u_i : X := \prod_{j \in N} X_j \to \mathbb{R}$ is a von Neumann-Morgenstern utility function which assigns to each strategy profile $x = (x_1, ..., x_n) \in X$ a payoff $u_i(x)$.

The various classes of games we are going to discuss are identified by means of different sets of properties concerning the structure of the strategic interaction. In particular, KLW assume that the following assumptions (P1)-(P4) are satisfied

(P1) There exists a finite set F such that $X_i = F$ for all players $i \in N$.

We call the set F the "facility set" and a strategy for player i is choosing an element out of F.

(P2) For each strategy profile $x \in X$ and all players $i, j \in N$:
if $x_i \neq x_j$ and $x'_j \in X_j$ such that $x_i \neq x'_j$, then $u_i(x_j, x_{-j}) = u_i(x'_j, x_{-j})$.

KLW call this assumption *independence of irrelevant choices* and the meaning is that for each player $i \in N$ and each strategy profile x the utility of i will not be altered if the set of players that choose the same facility is not modified.

Denote now by $n_f(x)$ the number of users of facility f in the strategy profile x. Then the third property defined by KLW can be stated as follows

(P3) For each player $i \in N$ and all strategy profiles $x, y \in X$, with $x_i = y_i$:
if $n_g(x) = n_g(y)$ for all $g \in F$, then $u_i(x) = u_i(y)$.

This *anonymity* condition reflects the idea that the payoff of player i depends on the number of players choosing the facilities, rather than on their identity. The fourth assumption, called *partial rivalry*, states that each player i would not regret that other players, choosing the same facility, would select another one. Formally:

(P4) For each player $i \in N$, each strategy profile $x \in X$, each player $j \neq i$ such that $x_j = x_i$ and each $x'_j \neq x_i$, $u_i(x_j, x_{-j}) \leq u_i(x'_j, x_{-j})$.

Though Milchtaich (1994) introduces his model in a slightly different way, the resulting class of games is essentially the same. More specifically he introduces the conditions (P1), (P4) and the following further assumption

(P2') For each player $i \in N$ and all profiles x, y with $x_i = y_i = f$, if $n_f(x) = n_f(y)$, then $u_i(y) = u_i(x)$.

In other words the utility of player i depends only on the number of users of the same facility he has chosen. It is straightforward to prove (assuming (P1)) that (P2') implies both (P2) and (P3), while we show that the reverse implication is true in the following

Lemma 2.1 . Any game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ satisfying (P1), (P2) and (P3) satisfies (P1) and (P2').

Proof: Take a game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ satifying (P1), (P2) and (P3). Let $i \in N$, $x, y \in X$ such that $x_i = y_i = f$ and furthermore assume that $n_f(x) = n_f(y)$. According to (P2) we know that for a fixed $x_k \neq x_i$, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i})$ where

$$(x'_j)_{j\in N\setminus i} = \begin{cases} x_i & \text{if } x_j = x_i \\ x_k & \text{otherwise} \end{cases}$$

and that $u_i(x_i, y_{-i}) = u_i(x_i, y'_{-i})$, where

$$(y'_j)_{j\in N\setminus i} = \begin{cases} x_i & \text{if } y_j = x_i \\ x_k & \text{otherwise} \end{cases}$$

Notice that for each $g \in F$, $n_g(x_i, x'_{-i}) = n_g(x_i, y'_{-i})$ and (P3) implies $u_i(x_i, x'_{-i}) = u_i(x_i, y'_{-i})$. Therefore, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i}) = u_i(x_i, y'_{-i}) = u_i(y_i, y_{-i})$. $\square$

KLW and Milchtaich independently proved the following

Theorem 2.1 Each normal form game $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ satisfying (P1), (P2), (P3) and (P4), possesses a pure strategy Nash equilibrium.

This is already a quite interesting result, but KLW went even further, showing

Theorem 2.2 For each game satisfying (P1), (P2) (P3) and (P4), the set of strong Nash equilibria is non empty.

Recall that, given a normal form game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a strategy profile x is called a strong Nash equilibrium if for every $S \subseteq N$ and all strategy profiles y_S, there is at least one player $i \in S$ such that $u_i(y_S, x_{-S}) \le u_i(x)$. We denote the set of strong Nash equilibria of a given game G by $SNE(G)$. Likewise the set of Nash equilibria is written as $NE(G)$.
Notice that in general the existence of a strong Nash equilibrium is not guaranteed.
Finally, we mention the model introduced by Quint and Shubik (1994), where the assumption that all players have the same set of facilities is relaxed. This new condition is defined as follows

(P1') There exists a facility set F such that for each $i \in N$, $X_i \subseteq F$.

Similar to Lemma 2.1 one can prove that a game G satisfies (P1'), (P2) and (P3) if and only if it satisfies (P1') and (P2').
QS are able to show

Theorem 2.3 For all strategic games satisfying (P1'), (P2') and (P4) there exists a pure strategy Nash equilibrium.

All the classes we have considered till now lack the potential property. Rosenthal (1973) studied a family of strategic interactions characterized by a congestion effect, which turns out to be isomorphic to the class of potential games. Instead of discussing his model, we briefly mention some results out of the Monderer and Shapley papers. Let us recall some preliminary definitions and properties.

Definition 2.1 (Monderer and Shapley, (1993)).
A strategic form game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is said to be a potential game if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, $x_i, x_i' \in X_i$ and $x_{-i} \in \prod_{j\neq i} X_j$ it holds that $u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i})$.
The function P is called a potential for G.

In other words, the potential function measures the difference in utility induced by a unilateral deviation. If a game admits a potential function, the potential is unique up to a constant. Notice that a potential maximizing strategy profile is certainly a Nash equilibrium and therefore each finite potential game possesses a pure strategy Nash equilibrium. The set of strategy profiles maximizing the potential function of a given potential game G is denoted by $PM(G) = \{x \in X | P(x) \geq P(y)$ for all $y \in X\}$. For an axiomatic approach to the potential maximizer we refer to Potters, Peleg and Tijs (1994).

Consider now the following assumption

(P1") For each player $i \in N, X_i \in 2^F$, $X_i \neq \emptyset$.

It means that each player has to choose among strategies which are given by a subset of the original facility set.
A *symmetry* condition describes the fact that the payoffs are player-independent, i.e. players who encounter each other by choosing the same facility obtain the same utility.

(P5) For each player $i, j \in N$ and every strategy profile $x \in X$ such that $x_i = x_j$, $u_i(x) = u_j(x)$.

Monderer and Shapley (1993) proved

Theorem 2.4 Each finite potential game is isomorphic to a game that satisfies the properties (P1"), (P2), (P3) and (P5).

As can be easily seen by means of Prisoner's Dilemma, potential games do not in general possess a strong Nash equilibrium. Since we want to describe games that satisfy the potential property and have strong Nash equilibria, we restrict ourselves to strategic interactions that satisfy properties (P1), (P2), (P3), (P4) and (P5). We will denote the class of (congestion) games which satisfy (P1) up to (P5) by $\mathcal{C}$.

3 On the structure of the class $\mathcal{C}$

In the previous section we have defined the class $\mathcal{C}$. In this section we are going to analyse its structure. Let $\mathcal{C}(n)$ denote the subclass with n players. It will be shown that each game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle \in \mathcal{C}(n)$ can be identified with a finite set of vectors in $\mathbb{R}^n_+$, and that the subclass $\mathcal{C}(F,n)$, consisting of all games with fixed facility set F, is a finitely generated cone in $(\mathbb{R}^n_+)^F$. The vector notation of the games simplifies the proofs of the theorems on strong equilibria and the potential maximizer we are going to present in paragraph 4 and 5. A simple example at the end of this section shows the existence of strong equilibria in a specific game.

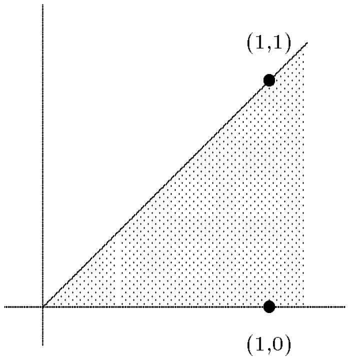
Fix a number $n \in \mathbb{N}$ and a finite set F. Recall that for a game $G \in \mathcal{C}(F,n)$ the utility functions are not player-specific and can be assumed to be induced by the functions w_f, $f \in F$, where $w_f : \{1,\ldots,n\} \to \mathbb{R}$ and $w_f(t)$ is the utility assigned to each player choosing facility f, if exactly t players choose it, for each $t \in \{1,\ldots,n\}$. Moreover, for each facility $f \in F$ the function $w_f : \{1,...,n\} \to \mathbb{R}$ is such that $w_f(t) \geq w_f(t+1)$ for all $t \in \{1,...,n-1\}$. For convenience and without loss of generality we assume that $w_f(t) \geq 0$ for all t. This means that the game $G \in \mathcal{C}(F,n)$ is described by $|F|$ vectors of the form $(w_f(1),...,w_f(n))$,$f \in F$, in the set $V = \{\mathbf{v} = (v_1,...,v_n) \in \mathbb{R}^n_+ | v_t \geq v_{t+1}$ for all $t \in \{1,\ldots,n-1\}\}$.

Proposition 3.1 The set V is a finitely generated cone in $\mathbb{R}^n_+$. The extreme directions of V are the vectors $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n$ with $\mathbf{b}^i = (\underbrace{1,1,1,1}_{i},...,0)$. Furthermore, $dim(V) = n$.

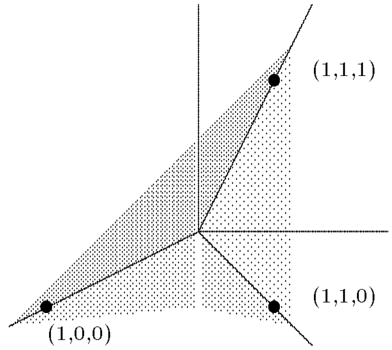
Proof: The vectors $\mathbf{b}^1 = (1,0,0,...,0)$, $\mathbf{b}^i = (\underbrace{1,1,1,1}_{i},...,0)$,..., $\mathbf{b}^n = (1,1,1,1,...,1)$ are elements of V and each vector $\mathbf{v} \in V$ can be uniquely written as a positive combination of $\mathbf{b}^1, \mathbf{b}^2, ..., \mathbf{b}^n$. To show this, let $\mathbf{v} \in V$ and define

$$\mathbf{B}_n = \begin{bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \vdots \\ \mathbf{b}^n \end{bmatrix}$$

Since $det(\mathbf{B}_n) = 1$, we see that the equation $\alpha \mathbf{B}_n = \mathbf{v}$ has exactly one solution for each $\mathbf{v} \in \mathbb{R}^n_+$. Clearly, α is positive because of the decreasingness property of $\mathbf{v}$. The set V is therefore the cone $C(\mathbf{B}_n)$ where $C(\mathbf{B}_n) := \{\alpha \mathbf{B}_n | \alpha \in \mathbb{R}^n_+\}$.



$C(\mathbf{B}_2)$



$C(\mathbf{B}_3)$

The extreme directions of the cone $C(\mathbf{B}_n)$ are the vectors $\mathbf{b}^i$, $i \in \{1, ..., n\}$. This cone has furthermore the property that its dimension is the number of extreme directions. In other words we have that $dim\ C(\mathbf{B}_n) = rank(\mathbf{B}_n) = n$. $\square$

Essentially we proved

Corollary 3.1 The class of games $\mathcal{C}(F, n)$ can be identified with a cone in $(\mathbb{R}^n_+)^F$ and $dim(\mathcal{C}(F, n)) = |F| \times n$.

In next example we are going to consider an extreme game of $\mathcal{C}(F, n)$, i.e. a game with facility set F such that w_f is an extreme direction in the cone V for each $f \in F$.

Example 3.1 Let G be a game in $\mathcal{C}(\{f, g\}, 4)$ such that $w_f = (1, 0, 0, 0)$ and $w_g = (1, 1, 0, 0)$.
Nash equilibria are either those strategy profiles in which one of the players chooses f and the other three g, or those in which both facilities are chosen by two players. These situations will be depicted

$$(\boxed{1}, 0, 0, 0)$$
$$(1, 1, \boxed{0}, 0)$$

for the first case and

$$(1, \boxed{0}, 0, 0)$$
$$(1, \boxed{1}, 0, 0)$$

for the second one, where the numbers in the square boxes indicate the payoff received by each player choosing this facility. Notice furthermore that the players are interchangeable as suggested by the symmetry condition (P5), which therefore guarantees that the Nash equilibria are also symmetric. One easily checks that all Nash equilibria are strong.

4 Strong Nash equilibria and the potential maximizer

In this section we prove our main theorem:

Theorem 4.1 On the class $\mathcal{C}$ of games, $SNE = NE = PM$.

A proof of this result is given in parts. Recall that for any game in normal form G, $SNE(G) \subseteq NE(G)$ and that for any potential game G, $PM(G) \subseteq NE(G)$. It is therefore sufficient to prove the following propositions.

Proposition 4.1 For each game $G \in \mathcal{C}$, $NE(G) \subseteq SNE(G)$.

Proposition 4.2 For each game $G \in \mathcal{C}$, $NE(G) \subseteq PM(G)$.

The proofs will be given using the structure of the class of games described in the previous section. We assume $n \in \mathbb{N}$ and a finite set F to be fixed. Each game $G \in \mathcal{C}(F, n)$ will be given by the vectors

$$\{(w_f(1), ..., w_f(n))\}_{f \in F}$$

A strategy combination will be written, irrespective of the specific players, in the same way as suggested in example 3.1. To each strategy profile $x \in F^N$ an *occupation number* $n_f(x)$ is related for each facility $f \in F$, which depicts the number of users of that facility w.r.t. x.

Proof:(Proposition 4.1).
Let $G \in \mathcal{C}(F, n)$ be given by $\{(w_f(1), \ldots, w_f(n))\}_{f \in F}$ and let $x \in NE(G)$. Suppose $S \subseteq N$ can strictly improve the payoff for all its members by switching to a strategy combination $y_S \in F^S$. Call the resulting strategy combination $y = (y_S, x_{N \setminus S})$. If $n_f(y) > n_f(x)$ for some $f \in F$, a player $i \in S$ exists such that $x_i = f$ and $y_i = g \neq f$. This implies $w_f(n_f(x)+1) \geq w_f(n_f(y)) > w_g(n_g(y))$ which contradicts the fact that x is a Nash equilibrium. So $n_f(x) = n_f(y)$ for all $f \in F$. Therefore every player in S chooses a new facility already chosen by a member of S and obtains a higher utility. Among

the utilities assigned to members of S there is a maximum, since S is finite. Any player in S rewarded with this maximum cannot get more in the new configuration. Hence a contradiction arises. Every Nash equilibrium is strong. $\square$

Based upon the switching argument the next lemma shows the similarities in utilities for different Nash equilibria.

Lemma 4.1 Let $G \in \mathcal{C}(F,n)$ be determined by $\{(w_f(1),\ldots,w_f(n))\}_{f\in F}$ and let x and y be two Nash equilibria for the game G.
For all f, $g \in F$ such that $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$:
$w_f(l) = w_f(n_f(y)) = w_g(n_g(x)) = w_g(m))$ for all l, m with $n_f(x)+1 \leq l \leq n_f(y)$ and $n_g(y)+1 \leq m \leq n_g(x)$.

Proof: Let f, g and l, m be as described. Both x, y are equilibria and therefore $w_f(n_f(y)) \geq w_g(n_g(y)+1) \geq w_g(m) \geq w_g(n_g(x)) \geq w_f(n_f(x)+1) \geq w_f(l) \geq w_f(n_f(y))$ $\square$

A potential is given by the function $P : X \to \mathbb{R}$, where for each strategy combination x, $P(x) = \sum_{f\in F}\sum_{l=1}^{n_f(x)} w_f(l)$. [2] Notice that to compute this potential it is necessary to add the utilities of the used facilities up to the number of users. This means that in each vector w_f all the first $n_f(x)$ numbers are added.

As a consequence it is clear that by n times choosing the facilities with highest remaining numbers, from left on, in the set of vectors

$$\{(w_f(1),...,w_f(n))\}_{f\in F}$$

a potential maximizing profile is found.

Example 4.1 Let $G \in \mathcal{C}(\{f,g,h\},4)$ such that

$$w_f = (4,3,2,1)$$
$$w_g = (5,2,1,0)$$
$$w_h = (1,1,0,0)$$

Clearly the potential maximizing strategy combinations are those $x \in F^N$ with $n_f(x) = 3$, $n_g(x) = 1$ and those with $n_f(x) = 2$, $n_g(x) = 2$. Notice that $P(x) = 14$ and that in fact all Nash equilibria are potential maximizing.

[2]This is a modification of the formula given in Monderer and Shapley (1993).

Proof:(Proposition 4.2).
It is sufficient to show that $P(x) = P(y)$ if x is a Nash equilibrium and y a potential maximizing strategy combination. Suppose that x is a Nash equilibrium and that y is a potential maximizing strategy combination. Facilities $f \in F$ such that $n_f(x) = n_f(y)$ add as much to $P(x)$ as to $P(y)$. Furthermore, by Lemma 4.1, if $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$ for certain $f, g \in F$ then $w_f(l) = w_f(n_f(y)) = w_g(n_g(x)) = w_g(m)$ for all $l \in \{n_f(x)+1, \ldots, n_f(y)\}$ and $m \in \{n_g(y)+1, \ldots, n_g(x)\}$ The total contribution of the facilities in the set $\{f \in F |\; n_f(x) \neq n_f(y)\}$ to the potentials $P(x)$ and $P(y)$ is apparently the same. □

In the last part of this section we discuss the existence of strictly strong Nash equilibria in the class $\mathcal{C}$ of games. Recall that given a game $\langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a strategy profile $x \in X$ is a *strictly strong Nash equilibrium* if for all coalitions $S \subseteq N$ and strategy combinations y_S, $u_i(y_S, x_{-S}) = u_i(x)$ for all $i \in S$ or $u_i(y_S, x_{-S}) < u_i(x)$ for at least one $i \in S$. Notice that the set of strictly strong Nash equilibria is a subset of the set of strong Nash equilibria. In particular the existence of a strong Nash equilibrium does not guarantee the existence of a strictly strong one as can be seen in the following counterexample.

Example 4.2 Consider the game $G \in \mathcal{C}(\{f, g\}, 3)$

$$w_f = (4, \boxed{2}, 0)$$
$$w_g = (\boxed{3}, 2, 1)$$

where, as usual, the squared numbers depict a (strong) Nash equilibrium payoff.
If the two players choosing f agree that one of them switches to g and the other one sticks to f, the utility will still be 2 for the switching one but increases from 2 to 4 for the remaining player. A similar argument holds for the strong Nash equilibria given by

$$(\boxed{4}, 2, 0)$$
$$(3, \boxed{2}, 1)$$

5 Relaxations of the model

The class $\mathcal{C}$ is characterized by properties (P1), (P2), (P3), (P4) and (P5). It is obvious that relaxations of those properties will have consequences on the results presented in section 4. In particular we discuss what are the consequences of replacing (P1) by (P1').

For the model of Konishi, Le Breton and Weber this relaxation was considered by Quint and Shubik.

First we make some remarks regarding strong equilibria and the potential maximizer in the class which results by not assuming the crowding effect property (P4).
Denote by $\mathcal{CP}$ the class of normal form games which satisfy the properties (P1), (P2), (P3) and (P5). Each n person game G in $\mathcal{CP}$ is a potential game and can be represented by a set of (arbitrary) vectors $\{(w_f(1), \ldots, w_f(n))\}_{f \in F}$. It is obvious that not for every game $G \in \mathcal{CP}$ strong Nash equilibria will exist. But even the existence of a strong Nash equilibrium for a game G does not guarantee that each Nash equilibrium is strong too, nor that the strong equilibrium is a potential maximizer. Next examples shows a game $G \in \mathcal{CP}$ such that $\emptyset \neq SNE(G) \subset NE(G)$ and $SNE(G) \cap PM(G) = \emptyset$.

Example 5.1 Let $G \in \mathcal{CP}$ be the following 3 person game with facilities f and g.

$$w_f = (4, 0, \boxed{5})$$

$$w_g = (4, 2, 0)$$

A Nash equilibrium is depicted. It is certainly the only strong Nash equilibrium (up to symmetry), but the maximal potential arises at the non strong equilibria which are given by

$$(\boxed{4}, 0, 5)$$
$$(4, \boxed{2}, 0)$$

Consider now the class of games $\mathcal{C}'$ constituted by (P1'),(P2),(P3),(P4) and (P5). The result on the existence of strong Nash equilibria is still valid and furthermore it can be shown that

Theorem 5.1 For all $G \in \mathcal{C}'$, $NE(G) = SNE(G)$.

Proof: The proof resembles that of Proposition 4.1 and will therefore be omitted. $\square$

In this class of games, the set of potential maximizing strategy combinations does not, in general, coincide with the set of Nash equilibria, as can be seen in the following counterexample.

Example 5.2 Consider the game $G \in \mathcal{C}'(\{f, g, h\}, 5)$. Assume that three players have strategy set $\{f, h\}$ and two $\{g, h\}$. The payoff vectors are

$$w_f = (4, 2, \boxed{1}, 0, 0)$$
$$w_g = (\boxed{3}, 2, 0, 0, 0)$$
$$w_h = (\boxed{2}, 1, 1, 0, 0)$$

where, as usual, the squared numbers depict a Nash equilibrium payoff. It represents strategy combinations in which the three players with strategy set $\{f, h\}$ all play f. Consider now the equilibrium payoff in which two of those three play f and the other plays h.

$$w_f = (4, \boxed{2}, 1, 0, 0)$$
$$w_g = (3, \boxed{2}, 0, 0, 0)$$
$$w_h = (\boxed{2}, 1, 1, 0, 0)$$

The potential associated to such Nash equilibrium is higher than the previous one.

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